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Product of Soluble Minimax Groups

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ABSTRACT: In this paper we show that if the group $G=A_1\ldots A_t$ be the product of finitely many pairwise permutable abelian minimax subgroups $A_1,\ldots,A_t$. Then G is a soluble minimax group.

***Keywords*:** minimax, minimal condition, maximal condition, finite residual group.

INTRODUCTION

In 1950 J.Szip (See 20) studied bout products of groups concerned finite groups. In 1961 O.H.Kegel (See 8) and in 1958 H.Wielandt (See 10) expressed the famous theorem, whose states the solubility of all finite products of two nilpotent groups .

In 1955 N.Itô (See 7) found an impressive and very satisfying theorem for arbitrary factorized groups. He proved that every product of two abelian groups is metabelian. Besides that, there were only a few isolated papers dealing with infinite factorized groups. P.M. Cohn (1956) (See 18) and L.Redei (1950)(See 19) considered products of cyclic groups, and around 1965 O.H.Kegel (See 23 & 24) looked at linear and locally finite factorized groups.

In 1968 N.F. Sesekin (See 16) proved that a product of two abelian subgroups with minimal condition satisfies also the minimal condition . He and Amberg independently obtained a similar result for the maximal condition around 1972 (See 17 & 1). Moreover, a little later the proved that a soluble product of two nilpotent subgroups with maximal condition likewise satisfies the maximal condition, and its Fitting subgroups inherits the factorization. Subsequently in his Habilitationsschrift (1973) he started a more systematic investigation of the following general question. Given a (soluble) product G of two subgroups A and B satisfying a certain finiteness condition $\mathfrak{X}$, when does G have the same finiteness condition $\mathfrak{X}$?(See 17)

For almost all finiteness conditions this question has meanwhile been solved. Roughly speaking, the answer is 'yes' for soluble (and even for soluble-by-finite) groups. This combines theorems of B. Amberg (See 1, 2, 3,4 and 6) , N.S. Chernikov (See 5), S. Franciosi, F. de Giovanni (See 3,6,25,26,27,28, and 29), O.H.Kegel (See 8), J.C.Lennox (See 12) , D.J.S. Robinson(See 9 and 14), J.E.
Roseblade(See 13), Y.P.Sysak(See 30,31,32 and 33), J.S.Wilson
(See 34), and D.I.Zaitsev(See 11 and 15).

Now, in this paper, we study the structure of soluble minimax groups, and the end we prove that if the group $G=A_1\ldots A_t$ be the product of finitely many pairwise permutable abelian minimax subgroups $A_1,\ldots,A_t$. Then G is a soluble minimax group.

For do this, in chapter 2 we express the elementary lemmas and Theorems and in chapter three we prove the main Theorem.

2. Preliminaries

(Elementary properties and Theorems.)

In this chapter we express the elementary Lemma and difinitons whose used in prove the Man Theorem in chapter

2.1.Lemma

Let the group G=AB be the product of two subgroups A and B. If x, y are elements of G, then $G=A^xB^y$. Moreover, there exists an element z of G such that $A^x=A^z$ and $B^y=B^z$.

Proof

Write xy^{-1}=ab with a in A and b in B. If $z=a^{-1}x$, then x=az and $y=b^{-1}z$, so that $A^x=A^z$ and $B^y=B^z$. It follows that G= $A^z B^z= A^x B^y$.

2.2. Difinition

Recall that a finite group is a D_π - groups if evety π - subgroup is contained in a Hall π - subgroup and any two Hall π - subgroups are conjugate.

2.3. Lemma

Let the finite group G=AB be the product of two subgroups A and B. If A,B, and G are D_π - group, for a set π of primes, then there exist Hall π -subgroups A_0 of A and B_0 of B such that A_0B_0 is a Hall π -subgroups of G.

Proof

Let A_1, B_1, and G_1 be Hall π -subgroups of A, B, and G, respectively. Since G is a D_π - group, there exist elements x and y such that A_1^x and B_1^y are both contained in G_1. It follows from Lemma 2.1 that $A^x=A^z$ and $B^y=B^z$ for some z in G. Thus $A_0=A_1^{xz^{-1}}$ and $B_0=B_1^{yz^{-1}}$ are Hall π -subgroups of A and B, respectively, which are both contained in $G_0=G_1^{z^{-1}}$. Clearly the order of $A_0\cap B_0$ is bounded by the maximum π -divisor n of the order of $A\cap B$ since $|G|=\frac{|A|.|B|}{|A\cap B|}$, It follows that $|G_0|=\frac{|A_0|.|B_0|}{n}\le\frac{|A_0|.|B_0|}{|A_0\cap B_0|}=|A_0B_0|$. Therefore $A_0B_0=G_0$ is a Hall π -subgroup of G.

2.4. Corollary

Let the finite group G=AB=AK=BK be the product of three nilpotent subgroups, A,B, and K, where K is normal in G. Then G is nilpotent .

Proof

See([4], corollary 1.3.5)

2.5. Theorem (See [7])

Let the group G=AB be the product of two abelian subgroups A and B. Then G is metabelian.

Proof :

Let a,a_1 be elements of A and b, b_1 elements of B. Write $b^{a_1}=a_2b_2$ and $a^{b_1}=b_3a_3$, whit a_2 ,a_3 in A and b_2,b_3 in B. Then

$$[a,b]^{b_1a_1}=[a,b^{a_1}]^{b_1}=[a,b_2]^{b_1}=[a^{b_1},b_2]=[a_3,b_2]$$

and

$$[a,b]^{b_1a_1}=[a^{b_1},b]^{a_1}=[a_3,b]^{a_1}=[a_3,b^{a_1}]=[a_3,b_2].$$

This proves that the commutators [a,b] and [a_1,b_1] commute. Since the factor group G/[A,B] is abelian, it follows that $G'=[a,b]$, and hence G' is abelian.

2.6. Difinition

Recall that the FC-centre of a group G is the subgroup of all elements of G with a finite number of conjugates. A group is an FC-group if it coincides with its FC-centre.

2.7.Lemma

Let the group G=AB be the product of two abelian subgroups A and B, and let S be a factorized subgroup of G. Then the centralizer $C_G(S)$ is factorized . Moreover, every term of the upper central series of G is factorized.

Proof

Since S is factorized, we have that S=$(A\cap S)(B\cap S)$. Let x=ab be an element of S, where a is in $A\cap S$ and b is in $B\cap S$ .If c=a_1b_1 is an element of $C_G(S)$, with a_1 in A and b_1 in B, it follows that.

$$[a_1,x]=[a_1,ab]=[a_1,b]=\left[cb_1^{-1},b\right]=[c,b]^{b_1^{-1}}=1.$$

Therefore a_1 belongs to $C_G(S)$, and $C_G(S)$ is factorized by Lemma 1.1.1 of [4]. In particular, the center of G is factorized. It follows from Lemma 1.1.2 of [4] that also every term of the upper central series of G is factorized.

2.8. Lemma

Let the group G=AB be the product of two subgroups A and B. If A_1, B_1, and F are the FC-centers of A, B, and C, respectively, then F=$A_1F\cap B_1F$. In particular, if A and B are FC-groups, the FC-centre of G is factorized subgroup.

Proof

Let x be an element of $A_1F\cap B_1F$, and write x=au where a is in A_1 and u is in F. Since the centralizers $C_A(a)$ and $C_A(u)$ have finite index in A, the index |A: $C_A(x)$| is also finite. Similarly, $C_B(x)$ has finite index in B. Therefore |G:<$C_A(x)$,$C_B(x)$>| is finite by Lemma 1.2.5 of [4]. It follows that $C_G(x)$ has finite index in G and hence x belongs to F. Thus F=$A_1F\cap B_1F$.

2.9.Lemma (See [7])

Let the finite non-trivial group G=AB be the product of two abelian subgroups A and B. Then there exists a non-trivial normal subgroup of G contained in A or B.

Proof

Assume that {1} is the only normal subgroup of G contained in A or B. By Lemma 2.7 have Z(G)=$(A\cap Z(G))(B\cap Z(G))=1$.The centralizer $C=C_G(A\cap C_G(G'))\, contains AG',$ and so is normal in G. Since $B\cap(AZ(C))\le Z(G)=1,$ it follows that $AZ(C)=A(B\cap AZ(C))=A.$This Z(G) is a normal subgroup of G contained in A, and so Z(G)=1. Since G' is abelian by Theorem 2.5, we have $A\cap G'\le A\cap C_G(G')\le Z(C)=1.$
Similarly $B\cap G'\le B\cap C_G(G')\le Z(C)=1.$The factorizer $X=X(G')$has the triple factorization $X=A^*B^*=A^*G'=B^*G'$,Where $A^*=A\cap BG'$ *and* $B^*=B\cap AG'$. Thus X is nilpotent by Corollary 2.4, so that $Z(X)=(A\cap Z(X))(B\cap Z(X))$
is not trivial. Hence there exists a non-trivial normal subgroup N of X contained in A or B. Suppose that N is contained in A. Since G' normalizes N, we have $[N,G']\le N\cap G'\le A\cap G'=1.$ Therefore we obtain the contradiction $N\le A\cap G_G(G')=1.$

2.10.Corrollary

Let the finite group G=$A_1\ldots A_t$ be the product of pairwise permutable nilpotent subgroups $A_1,\ldots,A_t$. Then G is soluble.

Proof

Let p be a prime, and for every i=1…,t let P_1 be the unique Sylow
p-complement of A_i. If $i\neq j$, the subgroup A_iA_j is soluble by Theorem 2.4.3 of [4]. Hence it follows from Lemma 2.3,that P_iP_j is a Sylow p-complement of A_iA_j. Thuse the subgroups $P_1,\ldots,P_t$ pairwise permute, and the product $P_1P_2\ldots P_t$ is a Sylow p-complement of G. Since G has a Sylow p-complement for every prime p, it is soluble.

2.11. Theorem (See 11 & 12):

If the soluble-by-finite group G=AB is the product of two polycyclic-by-finite subgroups A and B, then G is polycyclic-by-finite.

Proof

Assume that G it not polycyclic-by-finite. Then G contains an abelian normal section U/V which is either torsion-free or periodic and is not finitely generated. Clearly the factorizer of U/V in G/V is also a counterexample. Hence we may suppose that G has a triple factorization G=AB=AK=BK, Where K is an abelian normal subgroup of G which is either torsion-free or periodic. By Lemma 1.2.6(i) of [4] (See also [17]) the group G satisfies the maximal condition on normal subgroups, so that it contains a normal subgroup M which is maximal with respect to the condition that G/M is not polycyclic-by-finite. Thus it can be assumed that every proper factor group of G is polycylic-by-finite.

2.12. Theorem (See 15)

Let the soluble group G=AB be the product of two subgroups A and B with finite abelian section rank. If at least one of the factors A and B has an ascending normal series with central or periodic factors, then G also has finite abelian section rank.

Proof

See ([4],Theorem 4.6.10) .

2.13. Theorem (See 6)

Let the group G=AB=AK=BK be the product of three nilpotent subgroups A, B, and K, where K is normal in G. If K is minimax, then G is nilpotent.

Proof :

See (4, Theorem 6.3.4).

2.14.Theorem (See 6)

Let the group G=AB=AK=BK be the product of two subgroups A and B and a minimax normal subgroup K of G.
(i) if A,B, and K are locally nilpotent, then G is locally nilpotent.
(ii) If A, B, and K are hypercentral, then G is hypercentral.

Proof

See ([4],Theorem 6.3.7).

2.15. Lemma

Let the group G=AB be the product of two abelian subgroups A and B such that $A_G=B_G=1$. Then the following hold .

(i) $A\cap B=Z(G)=1.$

(ii) $A\cap C_G(G')=B\cap C_G(G')=1,$ and in particular $A\cap G'=B\cap G'=1.$

(iii) The factorizer $X=X(G')$ of G' does not have non-trivial normal subgroups which are contained in A or B, so that in particular Z(X)=1.

(iv) The FC-centre of G is trivial.

Proof

(i) They Lemma 2.7 we have that

$$Z(G)=(A\cap Z(G))(B\cap Z(G));\ A_GB_G=1.$$

Hence Z(G)=1. Moreover, $A\cap B$ in contained in Z(G) and so is also trivial.

(ii)This follows from the first part of the proof of Lemma 2.9.

(iii) Let N be a normal subgroup of X contained in A.Then G' normalizes N, so that by (ii)

$$[N,G']=N\cap G=A\cap G'=1$$

Therefore N is contained in $A\cap C_G(G')=1$

(iv) Let a be an element of $A\cap F$, where F is the FC-centre of G. Since G' is abelian by Theorem 2.5, the mapping $\varphi: x\mapsto[x,a]$ is a G epimorphism from G' onto $[G',a]$. Hence $C_{G'}(a)=ker\varphi$ is a normal subgroup of G, and the abelian groups $G'/C_{G'}(a)$ and $[G',a]$ are G isomorphic. The factorizer X=X(G') of G' has the triple factorization

X=A*B*=A*G'=B*G',
Where $A^* = A \cap BG'$ and $B^* = B \cap AG'$. As $G'/C_{G'}(a)$ is finite, it follows from Theorem 2.13 that $X/C_{G'}(a)$ is nilpotent. Therefore $[G',a]$ is contained in some term of the upper central series of X. Since Z(X)=1 by(iii), we have $[G',a]$=1 and so a belongs to $A \cap C_G(G')$. Thus a=1 by (ii), and hence $A \cap F = 1$. Similarly $B \cap F = 1$. It follows from Lemma 2.8 that $F = (A \cap F)(B \cap F) = 1$.

2.16 . Theorem (See [22])

Let the group G=AB$\neq 1$ be the product of two abelian subgroups A and B, at least one of which has finite section rank. Then there exists a non-trivial normal subgroup of G contained in A or B.

Proof

Assume that $A_G = B_G = 1,$ so that $A \cap G' = B \cap G' = 1.$ by Lemma 2.15(ii). The factorizer $X = X(G')$ has the triple factorization
$X = (A \cap BG')(B \cap AG') = (A \cap BG')G' = (B \cap AG')G',$
And its centre is trivial by Lemma 2.15(iii). The subgroups $A \cap BG'$ and $B \cap AG'$ are isomorphic, and hence both have finite section rank. By Theorem 2.12 the metabelian group X has finite abelian section rank, and hence is hypercentral by Theorem 2.14. In particular $Z(X) \neq 1,$ a contradiction.

3. Main Result

In this chapter by used the Lemmas and Theorems of chaper 2, we prove the Basic theorem of this paper as follows.

3.1. Main Theorem

Let the group G=$A_1 \ldots A_t$ be the product of finitely many pairwise permutable abelian minimax subgroups $A_1,\ldots,A_t$. Then G is a soluble minimax group.

Proof

Assume that the theorem is false, and let G= $A_1 \ldots A_t$ be a counterexample for which the sum $t + \sum_{i=1}^{t} m(A_i)$ is minimal. Suppose that there are indices i<j such that $D = A_i \cap A_j$ is infinite. Then

$$D^G = D^{A_1 \ldots A_t} = D^{A_1 \ldots A_{i-1} A_{i+1} \ldots A_{j-1} A_{j+1} \ldots A_t} \leq A_1 \ldots A_i \ldots A_{j-1} A_{j+1} \ldots A_t.$$

It follows that D^G is a soluble minimax group. On the other hand, the factor group $\overline{G} = G/D^G$ is also a soluble minimax group since $m(\overline{A}_i) < m(A_i)$. This contradiction shows that $A_i \cap A_i$ is finite if $i \neq j$.

Let J_i be the finite residual of A_i for every i=1,…,t. It follows from lemma 2.15 that $J_i J_j$ is the finite residual of the soluble minimax group A_iA_j, so that it is abelian . Hence $L = \langle J_1, \ldots, J_t \rangle$ is an abelian group satisfying the minimal condition. As $[A_i, J_j] \leq J_i J_j \leq L,$ the subgroup L is normal in G. Assume that $J_i \neq 1$ for some i. Then $m(A_i L/L) < m(A_i),$ and so G/L is a soluble minimax group. This contradiction proves that $J_i = 1$ for each i. In particular the maximum periodic normal subgroup E of A_1A_2 is finite. If A_1A_2=E, then the soluble minimax group by Corollary 2.10 Thus E is properly contained in A_1A_2, and by Theorem 2.16 we may suppose that A_1E/E contains a non-trivial normal subgroup N/E of
$A_1A_2/E=(A_1E/E)(A_2E/E)$.

As A_1A_2/E has no finite-non-trivial normal subgroups, N/E must be infinite . Moreover, the index $|N : N \cap A_1| = |A_1 N : A_1| \leq |A_1 E : A_1|$ is finite. If M is the core of $N \cap A_1$ in $A_1 A_2$, then N/M has finite exponent and hence is finite. Therefore M is an infinite normal subgroup of $A_1 A_2$ contained is A_1. Since

$M^G = M^{A_3 \dots A_t} \leq A_1 A_3 \dots A_t$, it follows that M^G is a soluble minimax group. As above, G/M^G is also a soluble minimax group since $m(A_1 M^G/M^G) < m(A1)$. This contradiction proves the theorem.

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